JOURNAL OF APPROXIMATION THEORY 15, 275–277 (1975)

Uniqueness in One-Sided Linear Chebyshev Approximation

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Communicated by Oved Shisha

Let X be a compact normal space and $C(X)$ the space of continuous functions on X. For $g \in C(X)$ define

$$\| g \| = \sup\{| g(x)|: x \in X\}.$$

Let $\{\phi_1, ..., \phi_n\}$ be a linearly independent subset of $C(X)$ and define

$$L(A, x) = \sum_{k=1}^{n} a_k \phi_k(x).$$

The problem of one-sided approximation is: for a given $f \in C(X)$ to find $L(A^*, .)$ minimizing $\|f - L(A, .)\|$ over all A with $L(A, .) \geqslant f$. Such an element $L(A^*, .)$ is called a best one-sided approximation to f.

We wish to find a necessary and sufficient condition for each $f \in C(X)$ to have a unique best approximation. It turns out that this condition is the classical Haar condition for uniqueness in ordinary Chebyshev approximation.

THEOREM. *A necessary and sufficient condition that each $f \in C(X)$ have a unique best approximation is that $\{\phi_1, ..., \phi_n\}$ be a Chebyshev set.*

Proof. Sufficiency. That $\{\phi_1, ..., \phi_n\}$ is a Chebyshev set implies that there is A with $L(A, .) > 0$. $L(A, .) > 0$ implies that for a given f there exists a μ such that $L(\mu A, .) > f$, and standard arguments then show the existence of a best approximation. Uniqueness is easily shown from known results [1; 2, pp. 446–450].

Necessity. If no $L(A, .) > 0$ exists, $f > 0$ has no best approximation. We will suppose that there is $L(A, .) > 0$. By arguments given above this implies the existence of a best approximation to all $f \in C(X)$. Let V_n be the linear space generated by $\{\phi_1, ..., \phi_n\}$. Suppose there exists $h_0 \in V_n$ which has distinct zeros $x_1, ..., x_n$ in X, then we shall show that there exists $f \in C(X)$ with more than one best approximation. Without loss of generality we can assume that $\| h_0 \| = 1$. By matrix theory it follows that we can find $B_1, \ldots, B_n$ not all zero such that

$$\sum_{i=1}^{n} B_i h(x_i) = 0 \qquad \textit{for every } h \in V_n .$$

We can assume without loss of generality that $B_j < 0$ for at least one j, $1 \leqslant j \leqslant n$. Let S be a continuous function on X such that $0 \leqslant S(x) \leqslant 1$ for all $x \in X$ and for $i = 1, \ldots, n$,

$$\begin{aligned} S(x_i) &= 0 \qquad \text{if} \quad B_i \geqslant 0, \\ &= 1 \qquad \text{if} \quad B_i < 0. \end{aligned}$$

Such a continuous function on X exists by Urysohn's lemma. Let

$$f(x) = \min\{-S(x)(1 - | h_0(x)|), h_0(x)\}$$

and let

$$\delta_n(f) = \inf\{\| f - h \|: h \in V_n \text{ and } f \leqslant h\}.$$

We shall show first that $\delta_n(f) \geqslant 1$. We have, for every $h \in V_n$, $f \leqslant h$,

$$\begin{aligned} \sum_{B_j<0} | B_j | &= \sum_{j=1}^{n} B_j f(x_j) \\ &= \sum_{j=1}^{n} B_j (f(x_j) - h(x_j)) \\ &= -\sum_{B_j<0} B_j(h(x_j) - f(x_j)) - \sum_{B_j \geqslant 0} B_j(h(x_j) - f(x_j)) \\ &\leqslant \delta_n(f) \sum_{B_j<0} | B_j |. \end{aligned}$$

This shows that $\delta_n(f) \geqslant 1$.

Next, from the definition of f it follows immediately that $\| f \| = 1$ and $\| f - h_0 \| = 1$.

Since $f \leqslant 0$ and $f \leqslant h_0$, any one of these two relations implies that $\delta_n(f) \leqslant 1$. Hence, we have $\delta_n(f) = 1$.

Finally, since $f \leqslant 0$ and $f \leqslant h_0$ and $\| f - 0 \| = \delta_n(f)$ and $\| f - h_0 \| = \delta_n(f)$, the function f has two distinct best one-sided approximations from V_n, namely 0 and h_0, and the theorem is completely proved.

The proof of necessity parallels that of Meinardus [3, pp. 17–18]. It was suggested by the referee as more appropriate than the original proof of the author.

References

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